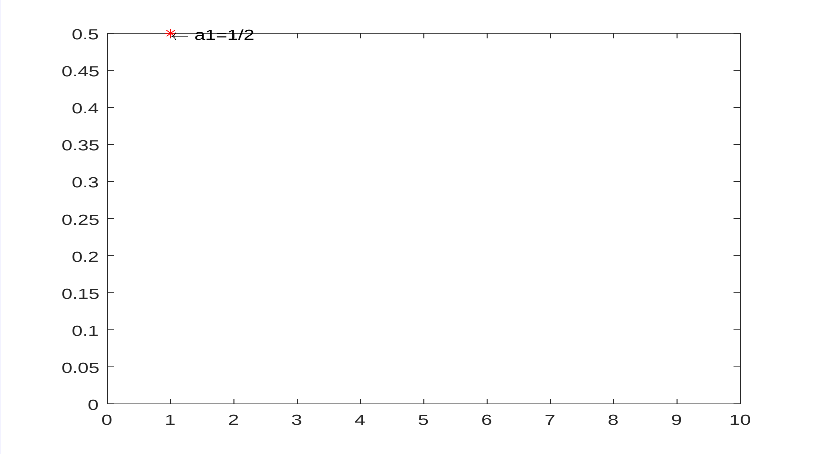
Funkcionalni nizovi

November 9, 2017

Nizovi realnih brojeva

$(a_n)_{n\in\mathbb{N}}, a_n = (\frac{1}{2})^n$

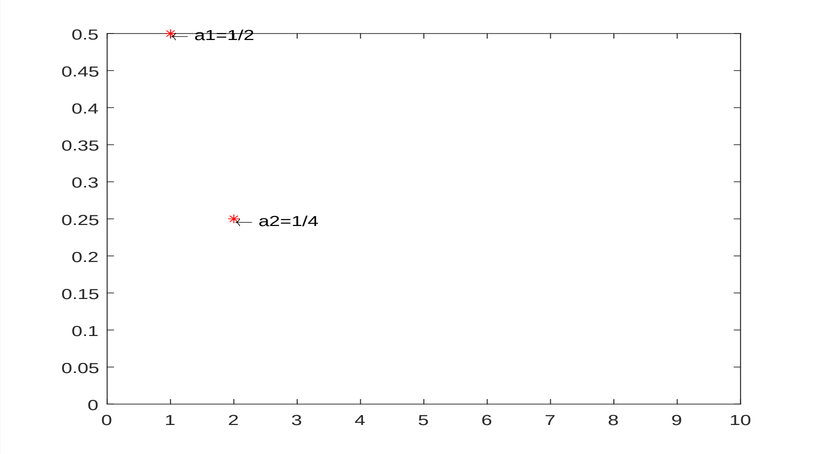
$a_1 = \frac{1}{2}$



Nizovi realnih brojeva

$(a_n)_{n\in\mathbb{N}}, a_n = (\frac{1}{2})^n$

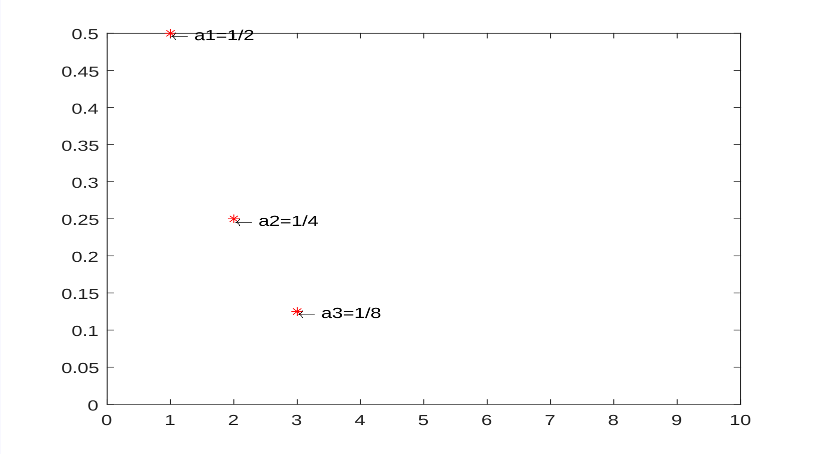
$a_1 = \frac{1}{2}, \qquad a_2 = \frac{1}{4}$



Nizovi realnih brojeva

$(a_n)_{n\in\mathbb{N}}, a_n = (\frac{1}{2})^n$

$a_1 = \frac{1}{2}, \qquad a_2 = \frac{1}{4} \qquad a_3 = \frac{1}{8}$

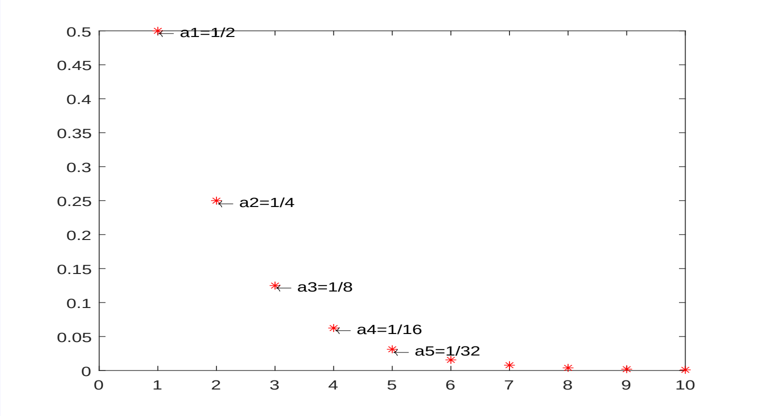


Nizovi realnih brojeva

$(a_n)_{n\in\mathbb{N}}, a_n = (\frac{1}{2})^n$

$a_1 = \frac{1}{2}, \qquad a_2 = \frac{1}{4} \qquad a_3 = \frac{1}{8}, \qquad a_4 = \frac{1}{16}, \qquad a_5 = \frac{1}{32}, \ldots.$

$\lim_{n\to\infty} a_n = 0$



Funkcionalni niz

$$(f_n(x))_{n\in\mathbb{N}} = (f_1(x), f_2(x), f_3(x), ...)$$

Članovi niza su **funkcije**, a ne brojevi.

Primer

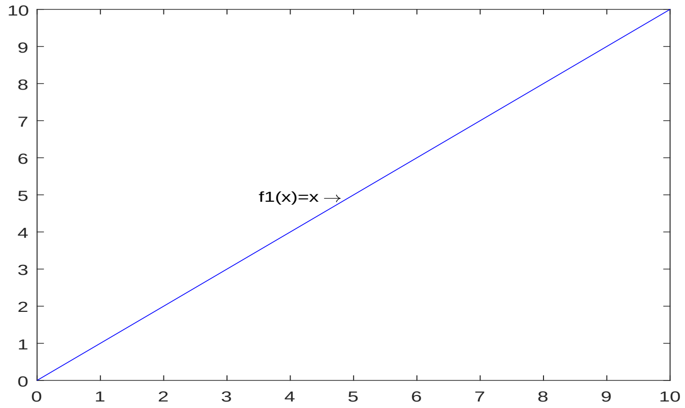
$f_1(x) = x, f_2(x) = \frac{x}{2}, f_3(x) = \frac{x}{3}, ...$

tj. $f_n(x) = \frac{x}{n}, \qquad (x \in \mathbb{R}, n \in \mathbb{N})$

Konvergencija funkcionalnig niza u tački

$f_n(x) = \frac{x}{n}, \qquad (x \in \mathbb{R}, n \in \mathbb{N})$

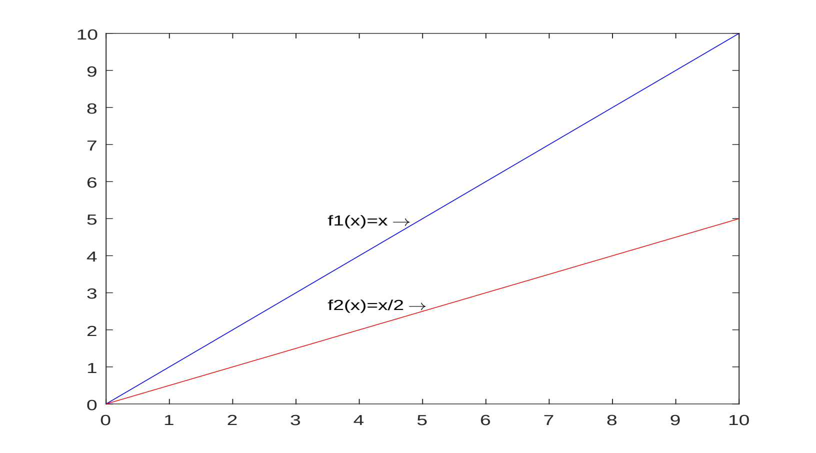
$f_1(x) = x$



Konvergencija funkcionalnig niza u tački

$$f_n(x) = \frac{x}{n}, \qquad (x \in \mathbb{R}, n \in \mathbb{N})$$

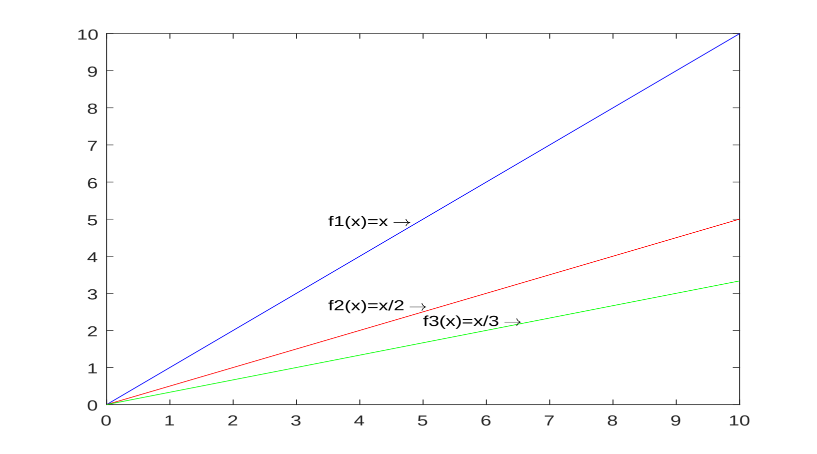
$$f_1(x) = x, \qquad f_2(x) = \frac{x}{2}$$



Konvergencija funkcionalnig niza u tački

$$f_n(x) = \frac{x}{n}, \qquad (x \in \mathbb{R}, n \in \mathbb{N})$$

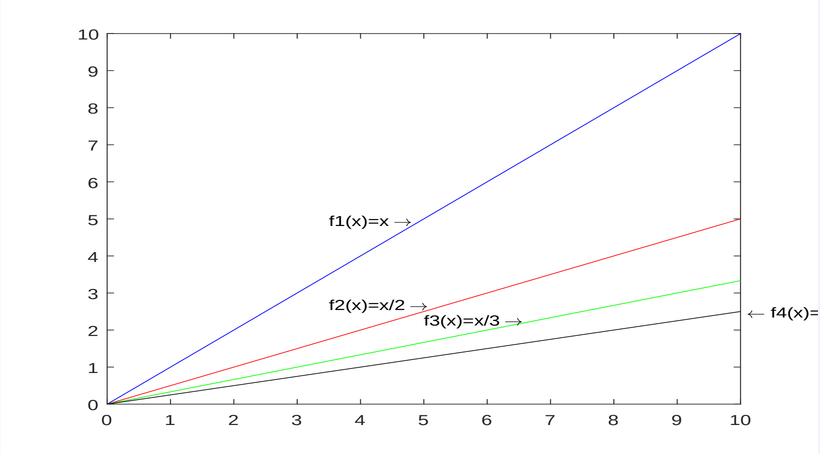
$$f_1(x) = x, \qquad f_2(x) = \frac{x}{2}, \qquad f_3(x) = \frac{x}{3}$$



Konvergencija funkcionalnig niza u tački

$$f_n(x) = \frac{x}{n}, \qquad (x \in \mathbb{R}, n \in \mathbb{N})$$

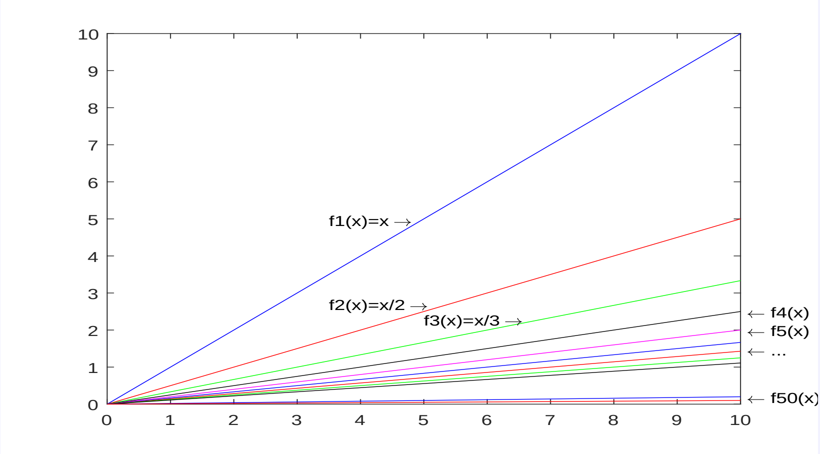
$$f_1(x) = x, \qquad f_2(x) = \frac{x}{2}, \qquad f_3(x) = \frac{x}{3}, \qquad f_4(x) = \frac{x}{4}, \ldots$$



Konvergencija funkcionalnig niza u tački

$$f_n(x) = \frac{x}{n}, \qquad (x \in \mathbb{R}, n \in \mathbb{N})$$

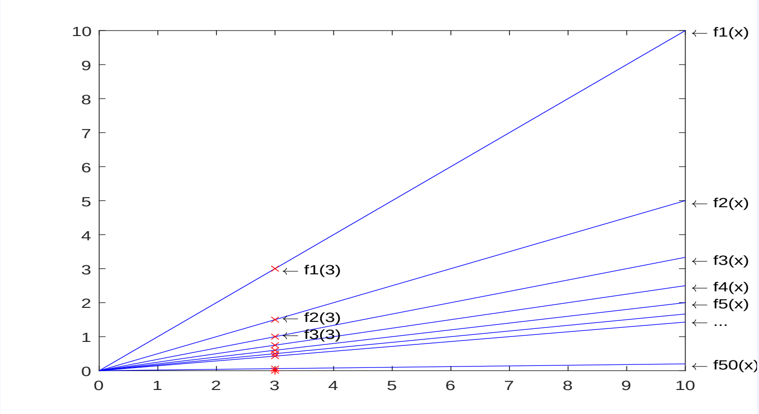
$$f_1(x) = x, f_2(x) = \frac{x}{2}, f_3(x) = \frac{x}{3}, f_4(x) = \frac{x}{4}, f_5(x) = \frac{x}{5}, \ldots, f_{50}(x) = \frac{x}{50}, \ldots$$



Konvergencija funkcionalnig niza u tački

$f_n(x) = \frac{x}{n}, \qquad (x \in \mathbb{R}, n \in \mathbb{N})$

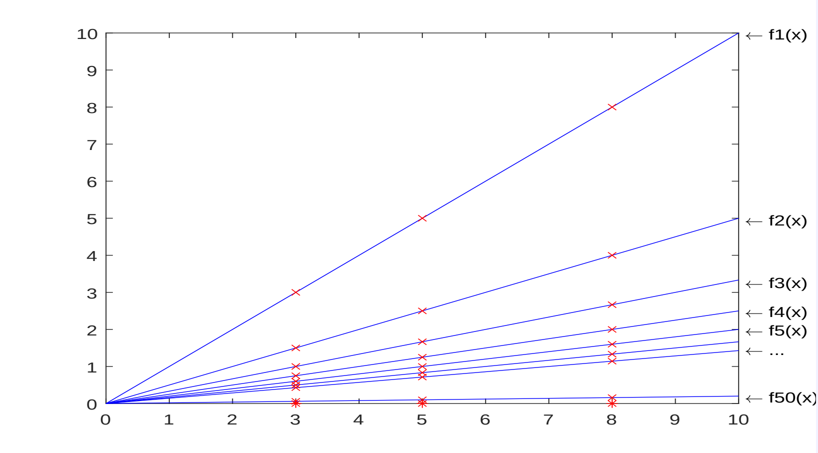
Za npr. $x = 3 : f_1(3) = 3, f_2(3) = \frac{3}{2}, f_3(3) = \frac{3}{3} = 1, f_4(3) = \frac{3}{4}, f_5(3) = \frac{3}{5}, ..., f_{50}(3) = \frac{3}{50}, ...$ dobija se brojni niz $(f_n(3))_{n\in\mathbb{N}}$ koji konvergira ka 0 $\Rightarrow$ funkcionalni niz $(f_n(x))_{n\in\mathbb{N}} = \frac{x}{n}$, konvergira u tački $x = 3$.



Konvergencija funkcionalnig niza u tački

$$f_n(x) = \frac{x}{n}, \qquad (x \in \mathbb{R}, n \in \mathbb{N})$$

Za $x = 3, x = 5, x = 8$:

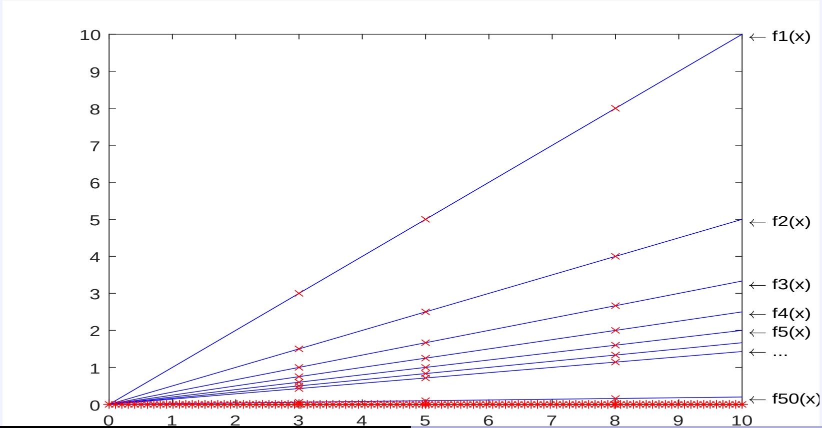


Konvergencija funkcionalnig niza na intervalu I

$$f_n(x) = \frac{x}{n}, \qquad (x \in \mathbb{R}, n \in \mathbb{N})$$

$$\lim_{n\to\infty} f_n(x) = f(x) = 0 \text{ (funkcija! a ne broj)}$$

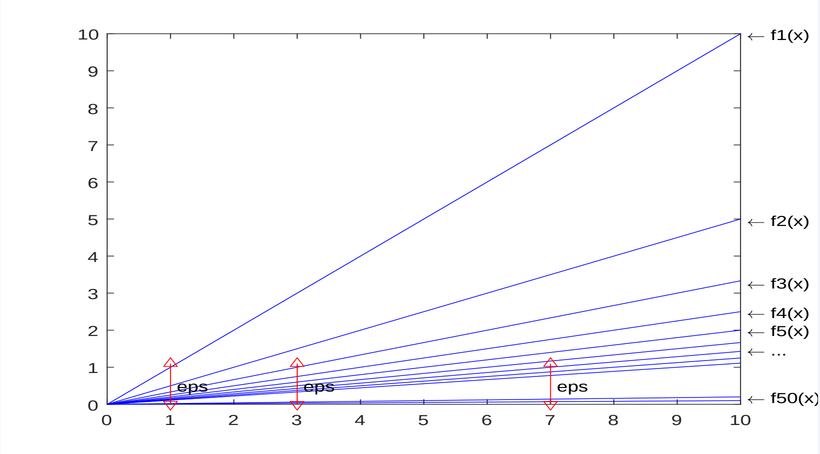
$$(\forall \varepsilon > 0)(\forall x \in I)(\exists N(\varepsilon, x))(\forall n > N(\varepsilon, x)) : |f_n(x) - f(x)| < \varepsilon$$



Konvergencija funkcionalnig niza na intervalu I

$$(\forall \varepsilon > 0)(\forall x \in I)(\exists N(\varepsilon, x))(\forall n > N(\varepsilon, x)) : |f_n(x) - f(x)| < \varepsilon$$

Za fiksirano (malo) ε, koliko veliko treba da bude n tako da važi $|f_n(x) - f(x)| < \varepsilon$?

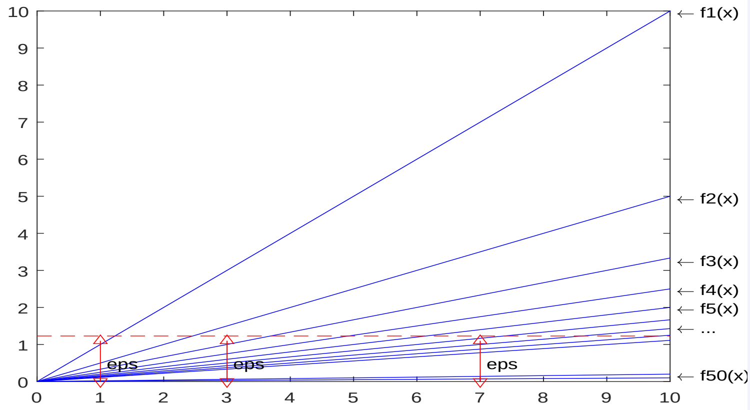


Konvergencija funkcionalnig niza na intervalu I

$$(\forall \varepsilon > 0)(\forall x \in I)(\exists N(\varepsilon, x))(\forall n > N(\varepsilon, x)) : |f_n(x) - f(x)| < \varepsilon$$

Odgovor zavisi od x!

Za $x = 1 : n \geq 1$. Za $x = 3 : n \geq 3$. Za $x = 7 : n \geq 6$.



Uniformna (ravnomerna) konvergencija funkcionalnig niza

$$(\forall \varepsilon > 0)(\forall x \in I)(\exists N(\varepsilon))(\forall n > N(\varepsilon)) : |f_n(x) - f(x)| < \varepsilon$$

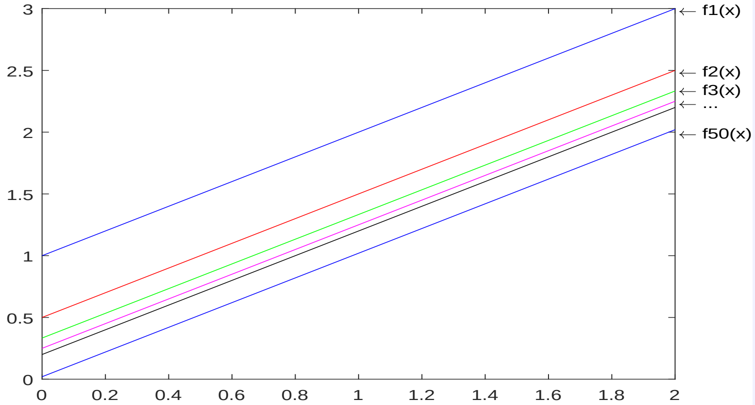
Primer

$$(f_n(x))_{n\in\mathbb{N}} = (x + \frac{1}{n})_{n\in\mathbb{N}} = (x + 1, x + \frac{1}{2}, x + \frac{1}{3}, ...)$$

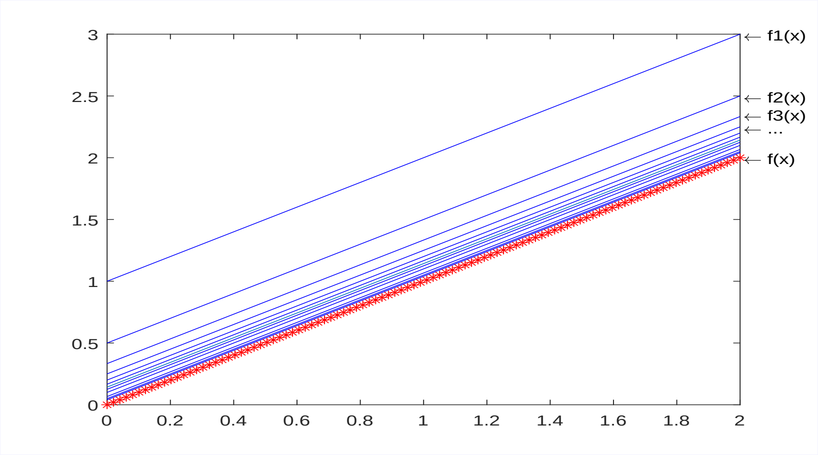
Uniformna (ravnomerna) konvergencija funkcionalnig niza

$(f_n(x))_{n\in\mathbb{N}} = (x + \frac{1}{n})_{n\in\mathbb{N}}$

$f_1(x) = x + 1, f_2(x) = x + \frac{1}{2}, f_3(x) = x + \frac{1}{3}, ..., f_{50}(x) = x + \frac{1}{50}, ...$



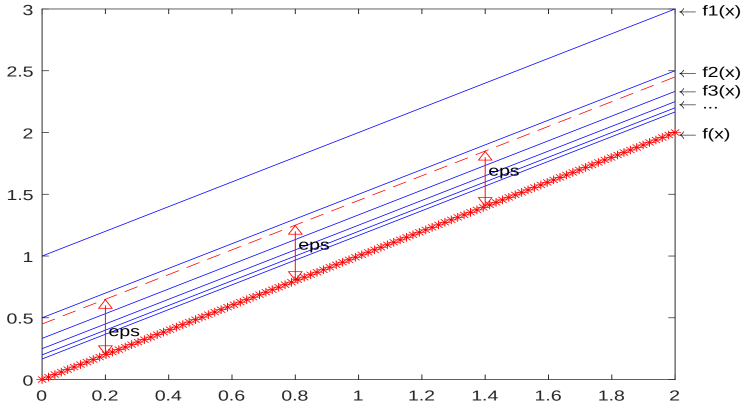
$(f_n(x))_{n\in\mathbb{N}} = (x + \frac{1}{n})_{n\in\mathbb{N}}$
$\lim\limits_{n\to\infty} f_n(x) = f(x) = x$ - funkcionalni niz konvergira ka $f(x) = x$.



Uniformna (ravnomerna) konvergencija funkcionalnig niza

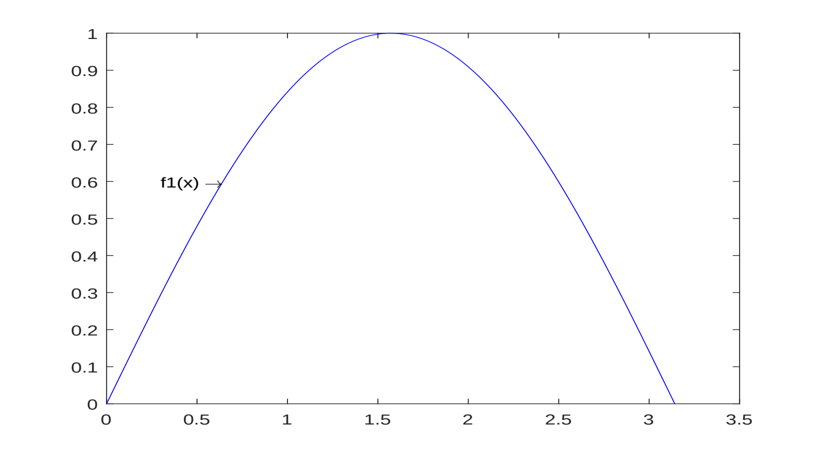
Za fiksirano (malo) ε, koliko veliko treba da bude n tako da važi $|f_n(x) - f(x)| < \varepsilon$?

Za $n \geq 3$. Odgovor ne zavisi od x! $\Rightarrow (f_n(x))_{n\in\mathbb{N}} \rightrightarrows f(x) = x$



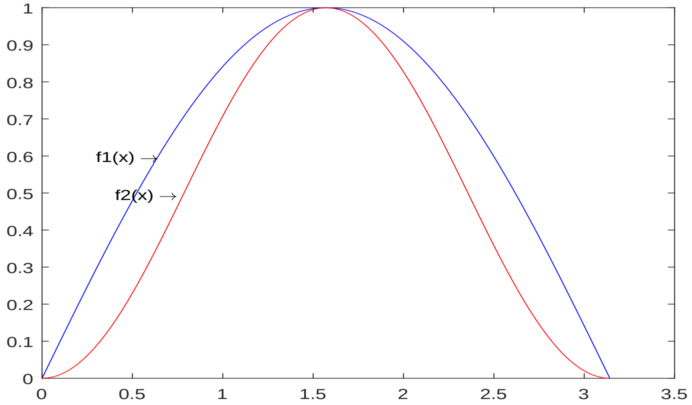
Još jedan primer

$(f_n(x))_{n\in\mathbb{N}} = (\sin^n(x))_{n\in\mathbb{N}},\ f_n : (0,\pi) \to \mathbb{R}$
$f_1(x) = \sin(x)$



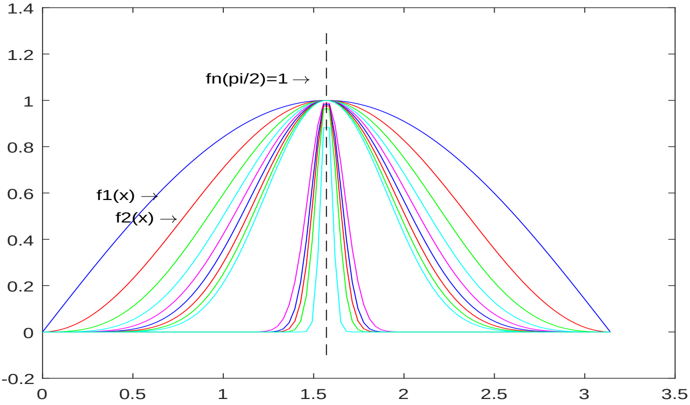
Još jedan primer

$(f_n(x))_{n\in\mathbb{N}} = (\sin^n(x))_{n\in\mathbb{N}},\ f_n : (0,\pi) \to \mathbb{R}$
$f_1(x) = \sin(x), f_2(x) = \sin^2(x)$



Još jedan primer

$(f_n(x))_{n\in\mathbb{N}} = (\sin^n(x))_{n\in\mathbb{N}}, \ f_n : (0,\pi) \to \mathbb{R}$
$f_1(x) = \sin(x), f_2(x) = \sin^2(x), f_3(x) = \sin^3(x), ..., f_{50}(x) = \sin^{50}(x), ...$



Još jedan primer

$(f_n(x))_{n\in\mathbb{N}} = (\sin^n(x))_{n\in\mathbb{N}},\ f_n : (0,\pi) \to \mathbb{R}$

$$\sin^n(x) \to \begin{cases} 0, za\ x \in (0,\pi/2) \cup (pi/2,\pi) \\ 1, za\ x = \pi/2 \end{cases}$$

Niz $(f_n(x))_{n\in\mathbb{N}}$ će da konvergira ka

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0, za\ x \in (0,\pi/2) \cup (pi/2,\pi) \\ 1, za\ x = \pi/2 \end{cases}$$

Granična funkcija $f(x)$ nije neprekidna, iako sve $f_n(x)$ jesu neprekidne
$\Rightarrow$ Niz $(f_n(x))_{n\in\mathbb{N}}$ ne konvergira uniformno.